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MOTION IN A RESISTING MEDIUM.

By James K. Whittemore.

§ 1. **Introduction.** Most treatises on the dynamics of a particle contain a discussion of the motion of a particle in a straight line under the action of forces expressed in terms of the velocity alone.* It may be remarked at the outset that the same discussion applies to motion in any path provided that the forces considered act along the tangent to the path. The simplest problem in which this condition is realized is motion in a resisting medium, where the force of resistance is opposed to the motion and increases in magnitude with the velocity. This problem is taken as typical in the following discussion. In such motion there is generally a "limiting velocity," sometimes attained, sometimes not attained by the moving particle. I have not found any treatment of the problem where it is shown how the actual velocity approaches the limiting velocity when that limit is not reached, or where the distance covered is compared with the distance which would be covered by a particle moving uniformly with the limiting velocity.† Indeed in so admirable a book as that of Appell, referred to above, there is, in the discussion of the motion of a heavy body falling in a resisting medium, this vague statement, "au bout d'un temps suffisamment longue le mouvement est sensiblement uniforme et de vitesse λ"; λ is the limiting velocity. This gap in the treatment of an important problem of elementary dynamics, I have attempted to fill in this paper. It is proved that if λ is a simple root of $F(v) = 0$, where $F(v)$ is the acceleration of the particle in its path, and when $F(v)$ is subject to certain other conditions generally satisfied in a real problem, $\lambda - v$ approaches zero at least as rapidly as e^{-mt} where t is the time and m a positive constant; that under the same conditions $\lambda t - x$, where x is the distance covered, approaches a limit L, and differs from L by a quantity not greater than a constant multiple of e^{-mt}. When λ is not a simple root of $F(v) = 0$ the results are different.

In § 3 I consider four problems in which there act forces exerted by constant powers, that is forces working at constant rates; such a force is inversely proportional to the velocity. That the expression for such a force C/v becomes infinite for a vanishing velocity does not invalidate the work, x and v remaining finite for finite t; of course in practice the power is not constant but is applied gradually. I have not found forces of this kind considered in any book on dynamics, yet they enter in real problems. It is certainly more reasonable to suppose that an engine works at a constant rate than to suppose, as is so often done, that it exerts a constant force.

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* See, for example, E. J. Routh, Dynamics of a Particle, Cambridge, 1898, pp. 51, 52, 56–59. P. Appell, Traité de mécanique rationelle, 2d ed., vol. 1, pp. 332–334.

† See however Appell, first edition, pp. 314–318, where these questions are considered for a falling body supposing resistance proportional to velocity.

§ 2. **Theory.** Consider a particle P moving with velocity v and tangential acceleration a, and suppose the force along the tangent to its path to depend on v alone, so that $a = F(v)$; let x be the distance covered in the time t, and suppose, for $t = 0$, $v = v_0 \geqq 0$. We assume that $F(v) = 0$ has a positive root λ, and further, if $v_0 < \lambda$, $F(v) = (\lambda - v)^p\varphi(v)$, $p > 0$, $\varphi(v) \geqq m > 0$, for $v_0 \leqq v \leqq \lambda$; $\varphi(\lambda)$ finite; if $v_0 > \lambda$, $F(v) = -(v - \lambda)^p\varphi(v)$, $p > 0$, $\varphi(v) \geqq m > 0$, for $v_0 \geqq v \geqq \lambda$; $\varphi(\lambda)$ finite. Evidently if p is an odd integer we may suppose $F(v)$ to have the same expression for values of v both greater and less than λ.

If $v_0 = \lambda$ we have $a = 0$; the motion is uniform and $x = \lambda t$. We consider in detail the case $v_0 < \lambda$. If $v_0 > \lambda$ we have similar results making the obvious changes in our statements.

Writing $a = dv/dt$, we have

$$t = \int_{v_0}^{v} \frac{dv}{(\lambda - v)^p\varphi(v)} \leqq \frac{1}{m}\int_{v_0}^{v} \frac{dv}{(\lambda - v)^p}, \qquad v_0 \leqq v \leqq \lambda. \tag{1}$$

If $p < 1$, $v = \lambda$ gives t a finite value; the particle attains the velocity λ in a finite time, thereafter moves uniformly with this velocity. If $p \geqq 1$, $v = \lambda$ in (1) makes t infinite. It follows that as t increases v approaches λ, always increasing, but not reaching that value in a finite time. For this reason λ is called the limiting velocity. If $p > 1$, equation (1) gives

$$t \leqq \frac{(\lambda - v)^{-p+1} - (\lambda - v_0)^{-p+1}}{m(p - 1)} < \frac{(\lambda - v)^{-p+1}}{m(p - 1)},$$
$$0 < (\lambda - v)^{p-1} < \frac{1}{mt(p - 1)}.$$

The case of most interest is that for which λ is a simple root of $F(v) = 0$, $p = 1$. For this case

$$t \leqq \frac{1}{m}\log\frac{\lambda - v_0}{\lambda - v}, \qquad 0 < \lambda - v \leqq (\lambda - v_0)e^{-mt}.$$

Writing now $a = vdv/dx$, we have

$$x = \int_{v_0}^{v} \frac{vdv}{(\lambda - v)^p\varphi(v)}. \tag{2}$$

Let P' be a particle moving uniformly with velocity λ in the same path as P, coinciding with P when $t = 0$; then PP', measured along the path, is equal to $\lambda t - x$. From (1) and (2)

$$\lambda t - x = \int_{v_0}^{v} \frac{dv}{(\lambda - v)^{p-1}\varphi(v)}.$$

If $p \geqq 2$, PP' increases indefinitely with t, that is as v approaches λ. If $p < 2$, PP' approaches a finite limit L as t increases, attaining this limit in a finite time only when $p < 1$. We have

$$PP' = \lambda t - x = L - \int_{v}^{\lambda} \frac{dv}{(\lambda - v)^{p-1}\varphi(v)}, \qquad L = \int_{v_0}^{\lambda} \frac{dv}{(\lambda - v)^{p-1}\varphi(v)}.$$

If $p = 1$,

$$0 < L - PP' = \int_{v}^{\lambda} \frac{dv}{\varphi(v)} \gtreqless \frac{\lambda - v}{m} \gtreqless \frac{\lambda - v_0}{m} e^{-mt}, \qquad L = \int_{v_0}^{\lambda} \frac{dv}{\varphi(v)}.$$

If $1 < p < 2$,

$$0 < L - PP' \gtreqless \frac{(\lambda - v)^{2-p}}{m(2 - p)} < \frac{1}{m(2 - p)[mt(p - 1)]^{(2-p)/(p-1)}}.$$

We may state the result of the preceding discussion as follows: for all cases in which $p < 2$ there is a "lost distance" L, lost in attaining full speed λ. In non-mathematical language we may say that after a time "sufficiently long" P is for all subsequent time "practically" at the fixed distance L behind P'.

As previously stated a similar discussion might be given for the case $v_0 > \lambda$: v decreases from v_0 and reaches or approaches the limiting velocity λ; the lost distance L is in this case negative and is really a distance gained, $-L$.

It is of interest to introduce the "time lost" in attaining full speed, $T = L/\lambda$. We do not mean that full speed is attained in the time T, for theoretically the full speed or limiting velocity is never reached unless $p < 1$, but rather we mean that a point of the path "sufficiently" remote from the starting point is reached by P "practically" at an interval T after it is reached by P'. Obviously if $v_0 > \lambda$, T is negative.

§ 3. **Examples.** We consider in this section six examples falling under the preceding discussion.

Example 1. The force is a resistance proportional to any positive power of the velocity.

$$a = -k_1 v^n, \qquad n, k_1 > 0.$$

We have $p = n$, $\lambda = 0$. To have motion we must have $v_0 > 0$. The velocity becomes equal to λ, that is vanishes, in a finite time only if $n < 1$. The particle P' is at rest. There is a finite lost distance, $L =$

$-x$, here negative, only if $n < 2$. This motion is illustrated by a boat drifting in still water with no power applied.

Example 2. There is a constant accelerating force and a resistance proportional to any positive power of the velocity.

$$a = k - k_1 v^n, \qquad n,\ k,\ k_1 > 0.$$

Here $p = 1$, and λ is given by $k = k_1 \lambda^n$. The velocity approaches but does not reach the value λ, and there is a finite L. We have

$$F(v) = k_1(\lambda^n - v^n), \qquad \varphi(v) = k_1 \frac{\lambda^n - v^n}{\lambda - v}.$$

Writing $v = \lambda z$, $1 > z \geqq 0$,

$$\varphi(v) = k_1 \lambda^{n-1} \frac{1 - z^n}{1 - z} \geqq k_1 \lambda^{n-1} = m.$$

If, in particular, $v_0 = 0$, we have

$$L = \frac{\rho_n}{k_1 \lambda^{n-2}}, \qquad T = \frac{\rho_n}{k_1 \lambda^{n-1}}, \qquad \rho_n = \int_0^1 \frac{1 - z}{1 - z^n}\, dz.$$

It may be proved that, as n increases, ρ_n decreases from $\rho_0 = \infty$ to $\rho_\infty = .5$, $\rho_1 = 1$, $\rho_2 = .693$, $\rho_3 = .606$. Further, for $v_0 = 0$,

$$0 < \lambda - v \overset{\equiv}{<} \lambda e^{-mt} = \lambda e^{-\rho_n t/T} < \lambda e^{-t/2\tau},$$

$$0 < L - PP' \overset{\equiv}{<} \frac{\lambda}{m} e^{-mt} = \frac{L}{\rho_n} e^{-\rho_n t/T} < 2Le^{-t/2T}.$$

This case is illustrated by the motion of a body falling in a resisting medium under the action of gravity.

In the following examples we introduce the force exerted by a constant power.

Example 3. There is a constant accelerating power and a constant retarding force.

$$a = \frac{c}{v} - k, \qquad c,\ k > 0.$$

Here $p = 1$, $c = k\lambda$. The velocity approaches but does not become equal to λ, and there is a finite L. We consider a numerical case: the weight of a train, including locomotive, is 600 tons, the horse power of the locomotive is 500, frictional resistances 15 pounds per ton; the train starts from rest. Taking as units foot, pound, second, $g = 32$,

$$600 \times 2000 \frac{a}{32} = 500 \times \frac{550}{v} - 600 \times 15.$$

We find $\lambda = 275/9$ feet per second, about 21 miles per hour;

$$\varphi(v) = \frac{6}{25v}, \quad m = \frac{6}{25\lambda} = \frac{54}{11 + 25^2}, \quad L = 1945 \text{ feet}, \quad T = 64 \text{ seconds}.$$

$$0 < \lambda - v \leqq \lambda e^{-mt}, \qquad 0 < L - PP' \leqq 3890 e^{-mt}.$$

Example 4. There is a constant accelerating power and a resistance proportional to any positive power of the velocity.

$$a = \frac{c}{v} - k_1 v^n, \qquad n, c, k_1 > 0.$$

For this example $p = 1$, and λ is given by $c = k_1\lambda^{n+1}$. The velocity approaches λ without reaching that value and there is a finite L. We have

$$F(v) = k_1 \frac{\lambda^{n+1} - v^{n+1}}{v}, \qquad \varphi(v) = k_1 \frac{\lambda^{n+1} - v^{n+1}}{v(\lambda - v)}.$$

Setting $v = \lambda z$, $1 > z \geqq 0$,

$$\varphi(v) = k_1\lambda^{n-1}\frac{1 - z^{n+1}}{z(1 - z)} \geqq k_1\lambda^{n-1}\frac{1 - z^{n+1}}{1 - z} \geqq k_1\lambda^{n-1} = m.$$

If in particular $v_0 = 0$ we find

$$L = \frac{\sigma_n}{k_1\lambda^{n-2}}, \qquad T = \frac{\sigma_n}{k_1\lambda^{n-1}}, \qquad \sigma_n = \int_0^1 \frac{z(1 - z)}{1 - z^{n+1}}\,dz.$$

It may be proved that, for positive n, σ_n is a continuous function ot n, decreasing, as n increases, from $\sigma_0 = 1/2$ to $\sigma_\infty = 1/6$, $\sigma_1 = .307$, $\sigma_2 = .247006$, $\sigma_3 = .220$. We find, supposing always $v_0 = 0$,

$$0 < \lambda - v \leqq \lambda e^{-mt} = \lambda e^{-\sigma_n t/T} < \lambda e^{-t/6T},$$

$$0 < L - PP' \leqq \frac{L}{\sigma_n} e^{-\sigma_n t/T} < 6Le^{-t/6T}.$$

We compare the values of L and T in examples 2 and 4, constant force with resistance proportional to v^n, and constant power with resistance proportional to v^n, supposing in both cases $v_0 = 0$:

Example 2, $a = k - k_1 v^n$, $k = k_1\lambda^n$,

$$L = \frac{\rho_n}{k_1\lambda^{n-2}} = \frac{\lambda^2}{k}\rho_n = \frac{\rho_n}{k}\left(\frac{k}{k_1}\right)^{2/n}; \qquad \text{for } n = 2,\ L = \frac{.693}{k_1},$$

$$T = \frac{\rho_n}{k_1\lambda^{n-1}} = \frac{\lambda}{k}\rho_n = \frac{\rho_n}{k}\left(\frac{k}{k_1}\right)^{1/n}; \qquad \text{for } n = 1,\ T = \frac{1}{k_1}.$$

Example 4, $a = c/v - k_1 v^n$, $c = k_1\lambda^{n+1}$,

$$L = \frac{\sigma_n}{k_1\lambda^{n-2}} = \frac{\lambda^3}{c}\sigma_n = \frac{\sigma_n}{k_1}\left(\frac{k_1}{c}\right)^{(n-2)/(n+1)}; \qquad \text{for } n = 2,\ L = \frac{.247006}{k_1},$$

$$T = \frac{\sigma_n}{k_1\lambda^{n-1}} = \frac{\lambda^2}{c}\sigma_n = \frac{\sigma_n}{k_1}\left(\frac{k_1}{c}\right)^{(n-1)/(n+1)}; \qquad \text{for } n = 1,\ T = \frac{.307}{k_1}.$$

It is remarkable that in both these examples the lost distance L depends only on k_1 and is independent of the constant force or power for $n = 2$ and for no other value of n; that the same is true of the lost time T for $n = 1$.

Example 4 is illustrated by the motion of a steamer in still water driven by an engine working at a constant rate. We note also in this example the formula $c = \lambda^3\sigma_n/L$, from which it follows that, if for a particular ship L is constant for varying power, then the power required to give the ship velocity λ is proportional to λ^3 whatever the value of n.

In the following paragraphs we consider further the motion of a steamer, driven by an engine working at a constant rate, and supposed to start from rest. We speak of λ, the full speed, as if this velocity were actually attained. While that is not true theoretically it is true for practical purposes.

A formula for the indicated horsepower of the engine of a steamer, which is used by marine engineers, is

$$H = \frac{D^{2/3}V^3}{K},$$

where H is the indicated horsepower, D the weight or displacement in long tons, V the velocity in knots, that is in nautical miles of 6,080 feet each per hour, and K a constant depending on the shape and dimensions of the ship. Assuming that the motion comes under example 4 the equation of motion of the ship in foot, pound, second units, $g = 32$, is

$$2240D\frac{a}{32} = 550\frac{H}{v} - Rv^n,$$

where R is the resistance of the water in pounds for unit velocity. Comparing this equation with that of example 4

$$c = \frac{550 \times 32}{2240}\frac{H}{D}, \qquad k_1 = \frac{32}{2240}\frac{R}{D}, \qquad \lambda = \frac{6080}{3600}V.$$

From $c = k_1\lambda^{n+1}$,

$$H = \frac{R}{550}\left(\frac{6080}{3600}V\right)^{n+1} = \frac{D^{2/3}V^3}{K}.$$

It follows that $n = 2$. Since $\sigma_2 = 0.247006$,

$$L = 0.247006 \times \frac{2240}{32 \times 550} \times \left(\frac{6080}{3600}\right)^3 KD^{1/3} = 0.15144KD^{1/3},$$

where L is in feet. It is evident that for a given ship, that is for fixed K, with varying displacement D, L is proportional to $D^{1/3}$. For different ships of equal displacement L and K are proportional. Since for given D and V the indicated horsepower H varies inversely as K it is evident that the greater K the better is the design of the ship for economy of power. It follows that the greater the distance L lost in acquiring full speed the better is the design. The designs of ships of equal displacement may be compared by measurement of L, or, what amounts to the same thing, K may be computed with known D and L from the last of the preceding formulas. If it is not assumed that $n = 2$, and if the values of c, λ, L are determined σ_n may be found from the equation $c = \lambda^3\sigma_n/L$, then n found from σ_n and the law of resistance determined.

The lost distance L may easily be measured, for L is the difference in the distances run in any period of time τ at full speed and in the period τ from the start, provided that full speed is acquired in the period t from the start. Instead of thus measuring L the time T lost in acquiring full speed may be measured, and L found from $L = \lambda T$. The lost time T is the difference of the times of running ν miles at full speed and of running the first v miles, provided that full speed is acquired in the first ν miles.

We give finally the numerical value of L computed from

$$L = 0.15144KD^{1/3}$$

for five ships, the values of K and D being taken from C. W. Dyson's Practical Marine Engineering, 7th edition, pp. 616, 617.

1. Yacht,	$D =$ 366,	$K =$ 200,	$L =$ 216.7 feet
2. Small ship,	3,200	230	513.3
3. Ship,	7,243	240	703.2
4. Liner,	15,400	250	942.0
5. Tramp,	8,320	265	813.2.

Example 5. Suppose P a particle of unit mass, moving initially with velocity $v_0 = 2$, acted upon by a unit retarding power, a constant accelerating force 2, and a resistance v. While this is a highly artificial example it is one not impossible to realize physically. It is of a type quite different from the preceding. We have

$$a = -\frac{1}{v} + 2 - v,$$

giving $p = 2$, $\lambda = 1$, $\varphi(v) = 1/v$, $m = 1/2$. From § 2 we have

$$0 < v - 1 < \frac{2}{t},$$

while $P'P$ becomes infinite with t. Carrying out the integration

$$t = \frac{2 - v}{v - 1} - \log (v - 1),$$

from which it may be proved

$$0 < v - 1 \lesseqgtr \frac{1 + e^{-2}}{t} < \frac{1.143}{t}.$$

For the distance covered

$$x = \frac{1}{v - 1} - v + 1 - 2 \log (v - 1),$$

and

$$PP' = \lambda t - x = t - x = v - 2 + \log (v - 1).$$

Clearly PP' becomes negatively infinite as t increases indefinitely.

Example 6. The only force is a constant accelerating power, and the particle starts from rest.

We prove this fact: the average velocity measured from the start is always two-thirds of the actual velocity; conversely, if the average velocity of a moving particle measured from rest is always two-thirds of the actual velocity the accelerating force works at a constant rate.

We have $a = c/v$. There is no limiting velocity. Since, for $t = 0$, $x = v = 0$.

$$ct = \frac{v^2}{2}, \qquad cx = \frac{v^3}{3}, \qquad \frac{x}{t} = \frac{2}{3} v.$$

Conversely, if $3x = 2vt$, we have, differentiating with respect to t,

$$3v = 2(v + at),$$

from which

$$\frac{dv}{dt} = a = \frac{v}{2t}, \qquad \frac{v^2}{2} = ct, \qquad \frac{dv}{dt} = \frac{c}{v}.$$

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